Stresses and deformations in egg-shaped sludge digestors: discontinuity effects

Alphose Zingoni *

Department of Civil Engineering, University of Cape Town, Rondebosch 7701, Cape Town, South Africa

Received 30 October 2000; received in revised form 17 May 2001; accepted 23 May 2001

Abstract

In the first part of this study [J. Eng. Struct. 23 (2001) 1365–1372], detailed results for stresses and deformations throughout a hydrostatically-loaded egg-shaped sludge digestor were developed on the basis of the membrane hypothesis for shells. However, and as is well-known, such results are generally unreliable around the junctions of the shell components making up the digestor. A significant amount of bending and shearing will occur in the vicinity of these junctions. In this paper, a simplified but reasonably accurate approach for quantifying discontinuity effects is developed for general shells of revolution of a certain variation of thickness along the shell meridian, and applied to the present problem of the 3-region egg-shaped digestor shell, to yield explicit closed-form results for discontinuity, and hence net, stresses throughout the digestor. These results, being very general (all parameters are kept arbitrary), enable a quick and reasonably accurate estimate of stresses in egg-shaped digestors of the type considered, and will thus be useful to the structural analyst and designer. Any desired parametric studies may also be conducted on the basis of these analytical results. Numerical results are presented, enabling some observations to be made and design recommendations proposed for egg-shaped digestors. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Bending theory; Discontinuity effects; Shell theory; Shell of revolution; Egg-shaped sludge digestor

1. Introduction

Analytical results for stress resultants and deformations in hydrostatically loaded egg-shaped sludge digestors were derived in Part 1 of this study [1] on the basis of the membrane hypothesis for shells. A numerical example showed that membrane stresses are generally relatively small in the upper parts of the digestor, but rapidly increase in magnitude in the lower zones, where considerable hoop tension and meridional compression were noted to occur. At the junctions of the three shell regions, discontinuities in the membrane deformations (lateral displacement \( \delta \) and meridional rotation \( \nu \)) were noted, suggesting that considerable bending may occur around these junctions.

The purpose of this part of the investigation is to derive general closed-form results for the discontinuity stresses around the junctions of the egg-shaped digestor. Total stresses will then follow by superimposing the membrane stresses of Part 1 on these discontinuity stresses. This approach inherently assumes that the membrane solution is the particular integral of the governing differential equations for the shell, while the edge or discontinuity effects constitute the homogeneous solution (associated with the application of line edge loadings in the absence of surface loadings). This simplifying assumption is, of course, an approximation, but for the type of shells in question in which the applied loading and the geometrical parameters of the shell vary smoothly, continuously and ‘not too rapidly’ over the surface of the shell, the approach yields results which are exact for all practical purposes [2]. The paper ends by considering a numerical example, allowing some significant conclusions to be made.

Fig. 1 depicts a bending element of the axisymmetrically-loaded shell of revolution, with all actions relevant to the homogeneous problem shown: transverse shearing force \( Q_p \), in-plane stress resultants \( N_\phi \) and \( N_\nu \), and bending moments \( M_\phi \) and \( M_\nu \), all being per unit length of the applicable edge of the element. In problems of the
the order of 5%) for shell edges and junctions with revolution is adopted. This approximation has been based on the Geckeler approximation for general shells around the shell junctions, a simplification that follows:

\[ \begin{align*}
\left[ D \left( \frac{r_2}{r_1} \right) \sin \phi \right] \frac{d^3 V}{d\phi^3} + \left[ D \left( \frac{r_2}{r_1} \right) \cos \phi + \sin \phi \frac{d}{d\phi} \left( \frac{D_{r_2}}{r_1} \right) \right] \frac{dV}{d\phi} \\
+ \left[ \nu \left( \cos \phi \frac{dD}{d\phi} - D \sin \phi \right) - D \left( \frac{r_1}{r_2} \right) \cos^2 \phi \sin \phi \right] V = 0 \\
-r_1 r_2 (\sin \phi) Q_o \\
\end{align*} \]

where the material properties \( E \) (Young’s modulus) and \( v \) (Poisson’s ratio) are assumed to be constant throughout the shell, and the parameter \( D \) is the flexural rigidity of the shell, defined as follows:

\[ D = \frac{E t^3}{12(1-v^2)} \]  

2. Approximate bending theory for general shells of revolution

The Geckeler approximation depends on the fact that in thin shells of revolution which are nonshallow, terms in lower derivatives of \( V \) or \( Q_o \) may be neglected in comparison with terms in higher derivatives, owing to the rapidly decaying character of the edge effects as one moves away from the edge. Thus, the Reissner-Meissner pair of equations reduces to

\[ \frac{d^3 V}{d\phi^3} = -\frac{r_1^2}{D} Q_o \\
\frac{d^2 Q_o}{d\phi^2} = -\frac{r_1^2}{r_2^2} E t V 
\]

Use can be made of Eq. (3b) to eliminate \( V \) from Eq. (3a), which, after Geckeler-type simplifications, becomes

\[ \frac{d^4 Q_o}{d\phi^4} + 4\lambda^4 Q_o = 0 \]

where \( \lambda \) is the shell-slenderness parameter, defined as follows:

\[ \lambda = 3(1-v^2) \frac{r_1}{r_2 t^2} \]

axisymmetric deformation of shells of revolution, and as indicated in Fig. 1, these actions vary with respect to the meridional (\( \phi \)) direction, but are constant with respect to the hoop (\( \theta \)) direction.

As a means for quantifying the discontinuity effects around the shell junctions, a simplified bending theory based on the Geckeler approximation for general shells of revolution is adopted. This approximation has been shown to be reasonably accurate (errors do not exceed the order of 5%) for shell edges and junctions with meridional locations in the range \( 40^o \leq \phi_e \leq 140^o \) and ratios of shell thickness to minimum radius of curvature in the range \( t/r_{\text{min}} \leq 40 \), conditions which are fulfilled for egg-shaped thin concrete digestors of the geometric assembly in question. As the derivational steps for various quantities of interest are not readily available in the literature for general shells of revolution (they are readily available for the special case of spherical shells), these will be given here, which will also allow an appreciation of the approximate nature of the final results to be gained. The starting point will be the Reissner-Meissner pair of second-order differential equations in the meridional rotation \( V \) and the transverse meridional shearing force \( Q_o \), for general shells of revolution subjected to axisymmetric loading conditions [4–7]:

**Fig. 1.** Element of an axisymmetrically-loaded shell of revolution under the action of transverse shearing force \( Q_o \), in-plane stress resultants \( N_t \) and \( N_o \), and bending moments \( M_t \) and \( M_o \).
In the spherical ends of the egg-shaped digestor shell, \(r_1=r_2=a\) (constant) and the shell thickness \(t\) is also constant. Therefore, the parameter \(\lambda\) will be constant in these regions. In the middle ogival part of the digestor shell, \(r_1=A\) (constant) and \(r_2=ah\) (constant), as already justified in Part 1. Therefore, the parameter \(\lambda\) will again be constant for the middle ogival shell, in which the shell thickness \(t\) is deliberately reduced as one moves away from the shell junctions. (In those cases where \(\lambda\) does vary, such as when the circular ogival shell is provided with constant thickness, the small variation of \(\lambda\) over the relatively narrow extent of the bending-disturbance zone allows the adoption of a constant average value of \(\lambda\) without incurring significant additional errors over those already inherent in the Geckeler approximation.)

The solution to Eq. (4) may be written as

\[
Q_{\phi} = C e^{-\lambda \psi} \sin(\lambda \psi + \beta) \tag{6}
\]

where \(C\) and \(\beta\) are constants of integration, and the coordinate \(\psi\) is the angle measured from the shell edge (Fig. 2). Denoting the meridional angle corresponding to the shell edge by \(\phi_e\), the angle \(\psi\) is given by \(\psi=\phi_e-\phi\) for shells with edges facing the direction of increasing \(\phi\), and by \(\psi=\phi-\phi_e\) for shells with edges facing the direction of decreasing \(\phi\) (see Fig. 2).

Stress resultants \(N_{\phi}^e\) and \(N_{\theta}^e\) associated with the bending disturbance or edge effect, bending moments \(M_{\phi}\) and \(M_{\theta}\), as well as deformations \(\delta^b\) and \(V^b\) of the edge effect, are then obtained on the basis of Eq. (6) using well-known relationships of the bending theory of axisymmetric shells of revolution [4–7], and/or using Geckeler-type simplifications, as follows (based on shells with edges facing the direction of increasing \(\phi\), that is, shells for which \(\psi=\phi_e-\phi\) so that \(\phi=\psi-\delta\psi\)):

\[
N_{\phi}^e = -Q_{\phi} \cot \phi = -\cot(\phi_e-\psi) C e^{-\lambda \psi} \sin(\lambda \psi + \beta) \tag{7a}
\]

\[
N_{\theta}^e = -\left( \frac{r_2}{r_1} \frac{dQ_{\phi}}{d\phi} - \frac{Q_{\phi}}{r_1} \frac{dr_2}{d\phi} \right) = -\frac{r_2}{r_1} \frac{dQ_{\phi}}{d\phi} \tag{7b}
\]

\[
\frac{2\lambda}{r_1} \sqrt{2} C e^{-\lambda \psi} \sin\left(\lambda \psi + \beta - \frac{\pi}{4}\right) \tag{7c}
\]

\[
M_{\phi} = -D \left( \frac{1}{r_1} \frac{dV}{d\phi} + \frac{\nu}{r_1} V \cot \phi \right) = -D \frac{dV}{d\phi} \tag{7d}
\]

\[
\frac{D}{E} \left[ \frac{r_2}{r_1} \frac{dQ_{\phi}}{d\phi} = \frac{1}{\lambda} \sqrt{2} \frac{r_1}{r_2} C e^{-\lambda \psi} \sin\left(\lambda \psi + \beta + \frac{\pi}{4}\right) \right] \tag{7e}
\]

\[
M_{\theta} = -D \left( \frac{1}{r_2} V \cot \phi + \frac{\nu}{r_1} \frac{dV}{d\phi} \right) = -\frac{1}{E} \frac{dV}{d\phi} = \nu \frac{M_{\theta}}{E} \tag{7f}
\]

\[
\delta^b = \frac{1}{E} r_2 \sin \phi (N_{\phi} - \nu N_{\theta}) = \frac{1}{E} (r_2 \sin \phi) N_{\theta} = \frac{1}{E} (r_2 \sin \phi) C e^{-\lambda \psi} \sin\left(\lambda \psi + \beta - \frac{\pi}{4}\right) \tag{7f}
\]

For the shell edges facing the direction of increasing \(\phi\) (that is, edges for which \(\phi=\phi_e-\psi\); see the portion to the left of the centreline in Fig. 2), the application of an edge bending moment \(M_{\phi}\) is associated with the boundary conditions

\[
(M_{\phi})_{\phi=\phi_e} = M_{\phi} \tag{8a}
\]

\[
(N_{\phi}^e)_{\phi=\phi_e} = 0 \tag{8b}
\]

yielding the constants

\[
C = \frac{2\lambda}{r_1} M_{\phi} \tag{9a}
\]

\[
\beta = 0 \tag{9b}
\]

while the application of an edge horizontal shearing force \(H_e\) is associated with the boundary conditions

---

Fig. 2. Shell-edge parameters \(\phi_e, \psi, M_{\phi}\), and \(H_e\) for shell edges facing the direction of increasing \(\phi\) (refer to left side of centreline) and shell edges facing the direction of decreasing \(\phi\) (refer to right side of centreline).
\[(M_{\phi})_{\phi=\phi_e}=0\]  
\[(N^h_{\phi=\phi_e}=-H_c\cos\phi_e\]  
yielding the constants \[C=-\sqrt{2}H_c\sin\phi_e\]  
\[\beta=-\frac{\pi}{4}\]  

Using the results for the constants \(C\) and \(\beta\) in Eqs. (7c) and (7f), for the application of \(M_e\) and \(H_e\) taken separately and then superimposing the contributions of \(M_e\) and \(H_e\), one obtains the edge deformations  

\[
\left[
\begin{array}{c}
V_e \\
\phi_e
\end{array}
\right] = 
\left[
\begin{array}{cc}
\frac{4\lambda^2}{EIr_1^2} & \frac{2\lambda^2}{Er_1^2}\sin\phi_e \\
\frac{2\lambda^2}{Er_1^2}\sin\phi_e & \frac{2\lambda^2}{Er_1^2}\sin^2\phi_e
\end{array}
\right] 
\left[
\begin{array}{c}
M_e \\
H_e
\end{array}
\right] 
\]  

(12)

The interior actions due to the simultaneous application of the edge actions \(M_e\) and \(H_e\) are obtained as  

\[N^b_0=-(\cot(\phi_e-\psi))e^{-\lambda\psi}\left[\frac{2\lambda}{r_1}M_e\sin\lambda\psi\right]\]  

\[-H_c(\sin\phi_e)(\sin\lambda\psi-\cos\lambda\psi)\]  

\[N^b_0=-2\left(\frac{r_1}{r_i}\right)\lambda e^{-\lambda\psi}\left[\frac{\lambda}{r_1}M_e(\sin\lambda\psi-\cos\lambda\psi)\right]\]  

\[+H_c(\sin\phi_e)\cos\lambda\psi\]  

\[M_o=e^{-\lambda\psi}\left[M_e(\sin\lambda\psi+\cos\lambda\psi)\right]\]  

\[+\frac{r_1}{\lambda}H_c(\sin\phi_e)\sin\lambda\psi\]  

\[M_o=\nu M_{\phi}\]  

(13)

For the shell edges facing the direction of decreasing \(\phi\) (that is, \(\phi=\phi_e+\psi\); see the portion to the right of the centre-line in Fig. 2), the simultaneous application of the edge actions \(M_e\) and \(H_e\), directed as shown on the right-hand side of the diagram in Fig. 2, leads to the edge deformations  

\[
\left[
\begin{array}{c}
V_e \\
\phi_e
\end{array}
\right] = 
\left[
\begin{array}{cc}
\frac{4\lambda^2}{EIr_1^2} & \frac{2\lambda^2}{Er_1^2}\sin\phi_e \\
\frac{2\lambda^2}{Er_1^2}\sin\phi_e & \frac{2\lambda^2}{Er_1^2}\sin^2\phi_e
\end{array}
\right] 
\left[
\begin{array}{c}
M_e \\
H_e
\end{array}
\right] 
\]  

(14)

and the interior actions  

\[N^b_0=-(\cot(\phi_e+\psi))e^{-\lambda\psi}\left[\frac{2\lambda}{r_1}M_e\sin\lambda\psi\right]\]  

\[+H_c(\sin\phi_e)(\sin\lambda\psi-\cos\lambda\psi)\]  

\[N^b_0=-2\left(\frac{r_1}{r_i}\right)\lambda e^{-\lambda\psi}\left[\frac{\lambda}{r_1}M_e(\sin\lambda\psi-\cos\lambda\psi)\right]\]  

\[-H_c(\sin\phi_e)\cos\lambda\psi\]  

\[M_o=e^{-\lambda\psi}\left[M_e(\sin\lambda\psi+\cos\lambda\psi)\right]\]  

\[+\frac{r_1}{\lambda}H_c(\sin\phi_e)\sin\lambda\psi\]  

\[M_o=\nu M_{\phi}\]  

(15)

In evaluating the above quantities, it must be noted that \(r_1=r_2=a\) for the two spherical ends, while for the middle ogival part, \(r_1=a\) and \(r_2=A-(b/\sin\phi)\).

For each of the three shell regions, once the redundant edge actions \(\{M_e, H_e\}\) have been evaluated, the interior actions \(N^b_0, N^b_0, M_o\) and \(M_o\) (associated with the edge bending disturbance) become known explicitly via the above sets of equations.

The net stresses in the shell, maximum on the inner and outer surfaces of the shell (where bending stresses are maximum), are given by superimposing the membrane stresses of Part 1 [1] with the edge effects of present considerations:

\[\sigma^m_\phi=\frac{N^b_0}{t}+\frac{N^b_0}{t}+\frac{6M_o}{t^2}\]  

(16a)

\[\sigma^m_\phi=\frac{N^b_0}{t}+\frac{N^b_0}{t}+\frac{6M_o}{t^2}\]  

(16b)

where the upper and lower signs \((\pm)\) associated with the \(M_o\) and \(M_o\) terms refer to the inner and outer surfaces of the shell, respectively.

3. Generalised results for the edge redundants \(M_e\) and \(H_e\)

In order to correct the noted mismatch in the membrane deformations \(\phi^m\) and \(\psi^m\) between adjacent sides of the junctions \(\phi=\phi_e\) and \(\phi=\phi_o\), it is assumed that axisymmetric bending moments and horizontal shearing forces act at the four edges of the separated shell regions, as shown in Fig. 3(a). Thus the bending moment \(M_i\) (per unit length) and horizontal shearing force \(H_i\) (per unit length) act over the (downward-facing) edge of the upper spherical cap, while \(M_i\) and \(H_i\) act over the (upward-facing) edge of the lower spherical cap; the action pairs
which the same parameters \( \{M_e, H_e\} \) were then applied at each edge as arbitrary actions, and the ensuing effects noted. It is now considered that \( M_e \) and \( H_e \) have specific values which in general differ for each edge, and which are dependent on the actual edge conditions (or, rather, inter-edge equilibrium and compatibility conditions).

Fig. 3(b) depicts the positive directions for the edge deformations \( \{\delta_1, V_1\}, \{\delta_2, V_2\}, \{\delta_3, V_3\} \) and \( \{\delta_4, V_4\} \) at the four edges of the separated shell regions; these deformations may be due to membrane actions, bending actions, or both.

In order to obtain results which are applicable to both the upper and the lower junctions of the three shell portions, consider a generalised junction (Fig. 4), for which \( \{M_u, H_u\} \) and \( \{M_l, H_l\} \) denote upper-edge and lower-edge corrective actions, respectively, at the junction. Thus, for the upper junction (located at \( \theta = \phi_o \)), \( \{M_u, H_u\} \) and \( \{M_l, H_l\} \) represent \( \{M_1, H_1\} \) and \( \{M_2, H_2\} \) respectively; for the lower junction (located at \( \theta = \phi_o \)), the same parameters represent \( \{M_3, H_3\} \) and \( \{M_4, H_4\} \) respectively. An element of the shell lying at the junction (refer to Fig. 4) would be kept in equilibrium by the reactive actions \( \{M_u, H_u, M_l, H_l\} \) and an additional action \( H_m^e \), which is the resultant towards the axis of revolution of the shell, of the membrane meridional stress resultants occurring at the shell edges, that is

\[
H_m^e = (N_m^u) - (N_m^l) \cos \phi_e \quad (17)
\]

Since \( (N_m^u) \) was previously [1] set equal to \( (N_m^l) \) in evaluating the constant of integration occurring in the general solution for \( N_m^u \) in the domain of the shell below the junction in question, it follows that \( H_m^e \) in Eq. (17) is equal to zero in this instance.
Continuity of deformations $V$ and $\delta$ across the junction between two adjacent regions of the shell requires that

$$
V'_i = V'_0 + V'_1 = V'_0 + V'_u + V'_l
$$

(18a)

$$
\delta'_i = \delta'_0 + \delta'_1 = \delta'_0 + \delta'_u + \delta'_l
$$

(18b)

where superscripts $m$, $b$ and $T$ denote membrane, bending-disturbance and total effects, respectively, while as before, subscripts $u$ and $l$ denote the upper and lower edges, respectively, of the junction in question.

For the bending-disturbance components of the net deformations, one may write

$$
\begin{bmatrix}
V'_u \\
\delta'_u
\end{bmatrix} =
\begin{bmatrix}
I_{11} & I_{12} \\
I_{21} & I_{22}
\end{bmatrix}
\begin{bmatrix}
M_u \\
H_u
\end{bmatrix}
$$

(19a)

$$
\begin{bmatrix}
V'_l \\
\delta'_l
\end{bmatrix} =
\begin{bmatrix}
J_{11} & J_{12} \\
J_{21} & J_{22}
\end{bmatrix}
\begin{bmatrix}
M_l \\
H_l
\end{bmatrix}
$$

(19b)

where $[I]$ and $[J]$ are the $2 \times 2$ matrices of influence coefficients applicable to the upper and lower edges, respectively, of the junction in question, as obtained earlier [Eqs. (12) and (14)]. This permits Eqs. (18a) and (18b) to be re-written as

$$
I_{11}M_u + I_{12}H_u + V'_0 = J_{11}M_u + J_{12}H_u + V'_u
$$

(20a)

$$
I_{21}M_u + I_{22}H_u + \delta'_0 = J_{21}M_u + J_{22}H_u + \delta'_u
$$

(20b)

Equilibrium of the element of the shell lying at the junction of adjacent shell regions (refer to Fig. 4) requires that

$$
M_u - M_l = 0
$$

(21a)

$$
H_u - H_l - H'_0 = 0
$$

(21b)

From the above conditions, and noting that $H'_0 = 0$ as already explained, the relationships

$$
M_l = M_u
$$

(22a)

$$
H_l = H_u
$$

(22b)

are obtained. Substituting for $M_l$ and $H_l$ in Eqs. (20a) and (20b), one obtains

$$
I_{11}M_u + I_{12}H_u + V'_0 = J_{11}M_u + J_{12}H_u + V'_u
$$

(23a)

$$
I_{21}M_u + I_{22}H_u + \delta'_0 = J_{21}M_u + J_{22}H_u + \delta'_u
$$

(23b)

which, upon rearranging, become

$$
(I_{11} - J_{11})M_u + (I_{12} - J_{12})H_u = V'_m - V'_0
$$

(24a)

$$
(I_{21} - J_{21})M_u + (I_{22} - J_{22})H_u = \delta'_m - \delta'_0
$$

(24b)

These simultaneous equations lead to the closed-form results

$$
M_u = \frac{(V'_m - V'_0)(J_{22} - J_{21}) - (\delta'_0 - \delta'_u)(I_{22} - J_{22})(I_{21} - J_{21})}{(I_{22} - J_{22})(I_{11} - J_{11}) - (I_{21} - J_{21})(I_{12} - J_{12})}
$$

(25a)

$$
H_u = \frac{(\delta'_m - \delta'_0)(I_{11} - J_{11}) - (V'_m - V'_0)(I_{12} - J_{21})}{(I_{22} - J_{22})(I_{11} - J_{11}) - (I_{21} - J_{21})(I_{12} - J_{12})}
$$

(25b)

Noting that $I_{21} = J_{12}$ and $J_{21} = J_{12}$, and defining the parameters

$$
\xi = V'_m - V'_0
$$

(26a)

$$
\mu = \delta'_m - \delta'_0
$$

(26b)

$$
K_1 = I_{11} - J_{11}
$$

(26c)

$$
K_2 = I_{12} - J_{12}
$$

(26d)

$$
K_3 = I_{22} - J_{22}
$$

(26e)

$$
K_4 = (I_{11} - J_{11})(I_{22} - J_{22}) - (I_{12} - J_{12})^2
$$

(26f)

allows the results to be expressed more compactly as follows:

$$
M_u = \frac{\xi K_2 - \mu K_3}{K_4} (= M_i)
$$

(27a)

$$
H_u = \frac{\xi K_2 - \mu K_3}{K_4} (= H_i)
$$

(27b)

### 4. Numerical results and observations

The example that was considered for membrane effects in Part 1 [1] is re-considered here for bending-disturbance and net effects.

For the slenderness parameter $\lambda$, Eq. (5) yields $\lambda = 9.2533$ for the spherical shell closures (with $v=0.15$, $r_i = r_o = a=5$ m, $t=0.1$ m) and $\lambda = 27.7598$ for the ogival middle shell [with $v=0.15$, $r_i = A=15$ m, $r_o = ah=5$ m(0.1 m)].

With $E=28 \times 10^9$ N/m$^2$, Eqs. (12) and (14) yield the following values for the influence coefficients:

(i) Upper junction ($\phi = \phi_o = 60^\circ$)

Upper edge of upper junction (spherical top closure)

$$
I_{11} = -2.2637 \times 10^7 \text{ N}^{-1}
$$

$$
I_{12} = 5.2966 \times 10^{-8} \text{ mN}^{-1}
$$

$$
I_{22} = -2.4786 \times 10^{-8} \text{ m}^2 \text{ N}^{-1}
$$

Lower edge of upper junction (upper end of ogival shell)

$$
J_{11} = 2.2637 \times 10^{-7} \text{ N}^{-1}
$$

$$
J_{12} = 5.2966 \times 10^{-8} \text{ mN}^{-1}
$$

$$
J_{22} = 2.4786 \times 10^{-8} \text{ m}^2 \text{ N}^{-1}
$$
Making use of the results of Part 1 for the membrane deformations at the edges of the three shell portions [1], the parameters \( \xi \) and \( \mu \) may readily be evaluated on the basis of Eqs. (26a) and (26b). The parameters \( K_1 \), \( K_2 \), \( K_3 \) and \( K_4 \) are evaluated on the basis of Eqs. (26c, 26d, 26e) and (26f), using the now-known values of the influence coefficients. The redundant corrective actions \( M_u \), \( H_u \), \( M_l \) and \( H_l \) readily follow from Eqs. (27a) and (27b). For the two junctions of the digestor, the obtained numerical values are summarised below.

(i) Upper junction \( (\phi=\phi_o=60^\circ) \)

\[
\xi = -4.2545 \times 10^{-5}
\]
\[\mu = 2.814 \times 10^{-5} \text{ m}\]

\[
K_1 = -4.5274 \times 10^{-7} \text{ N}^{-1}
\]
\[K_2 = 0\]

\[
K_3 = -4.9572 \times 10^{-8} \text{ m}^2 \text{ N}^{-1}
\]

\[
K_4 = 2.2443 \times 10^{-14} \text{ m}^2 \text{ N}^{-2}
\]

(ii) Lower junction \( (\phi=\phi_o=120^\circ) \)

\[
M_u = M_l = 93.9732 \text{ Nm/m}
\]
\[H_u = H_l = -567.6649 \text{ N/m}\]

\[
M_u = M_l = 943.0342 \text{ Nm/m}
\]
\[H_u = H_l = -5121.8948 \text{ N/m}\]
upwards from the supported level (\(\phi = 150^\circ\)) into the lower spherical closure is localised to the vicinity of \(\phi = 150^\circ\) and would have died out almost completely by the time the interior locations of \(\phi = 130^\circ - 140^\circ\) are reached.) The meridian was modelled as follows: From \(\phi = 0^\circ\) to \(\phi = 60^\circ\), and from \(\phi = 120^\circ\) to \(\phi = 150^\circ\), elements each subtending an angle of 1° at its centre of curvature were adopted. Between \(\phi = 60^\circ\) and \(\phi = 120^\circ\), elements each subtending an angle of \((1/3)^\circ\) at its centre of curvature were adopted. The mesh therefore had 270 elements and 541 nodes in total, each element being of arc length 87.27 mm.

Table 1 shows analytical results versus FEM (finite-element method) results for inner and outer-surface meridional and hoop stresses at the apex of the digester (\(\phi = 0^\circ\)), the equatorial location (\(\phi = 90^\circ\)) and the two discontinuity locations (\(\phi = 60^\circ\) and \(\phi = 120^\circ\)) of the shell. For the two shell junctions (\(\phi = 60^\circ\) and \(\phi = 120^\circ\)), superscripts + and − refer to the locations just above and just below the junction respectively. For all the six locations depicted in the table, upper-row values are the analytical results, while lower-row values (in square brackets) are the FEM results. It is evident that the obtained analytical results are in good agreement with the FEM results (discrepancy generally not exceeding 5% in the case of the higher and more critical values of stresses), showing that the closed-form analytical results derived in the present study are reliable. The stress plots and discussion that follow are based on the closed-form analytical results.

Figs. 5–8 are plots of meridional (\(\sigma_m\)) and hoop (\(\sigma_q\)) stresses, for the edge zone of the upper spherical closure, the upper edge zone of the ogival shell, the lower edge zone of the ogival shell, and the edge zone of the lower spherical closure, respectively. Each set of plots comprises the membrane-solution component \(\sigma^m\) (constant across the shell thickness), and the total-stress values \(\sigma^t(i)\) (for the inner surface of the shell) and \(\sigma^t(o)\) (for the outer surface of the shell). In presenting these plots, the main aim is to show the magnitude of the bending-disturbance contribution to the total-stress values, in relation to the membrane contribution. As the magnitude of the bending-disturbance contribution would have become insignificant beyond an angular distance of \(\psi = 20^\circ\) from the edges of the spherical closures and \(\psi = 10^\circ\) from the edges of the ogival portion, the plots show stress variations from the edges \(\psi = 0^\circ\) of the shells, up to only \(\psi = 20^\circ\) in the case of the spherical closures and \(\psi = 10^\circ\) (or less) in the case of the ogival ends.

Consider the meridional stress variations to begin with. In the upper spherical shell, meridional bending-disturbance stresses are significant in comparison with membrane stresses. The largest membrane meridional

---

**Table 1**

Comparison of analytical results (upper row) versus FEM results (lower row in square brackets) for the numerical example

| \(\phi\) (deg) | Location | Node no. | \(\sigma_m\) (N/mm²) | Inner | \(\sigma_q\) (N/mm²) | Inner | \(\sigma^t(i)\) (N/mm²) | Outer | \(\sigma^t(o)\) (N/mm²) | Outer |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0.00 | 0.00 | 0.00 | 0.00 |
| 60° | 121 | 0.33 | 0.22 | 1.09 | 1.07 |
| 60°+ | 121 | 0.33 | 0.21 | 1.09 | 1.07 |
| 90 | 301 | 1.28 | 1.28 | 7.35 | 7.35 |
| 120° | 481 | 1.93 | 3.06 | 10.64 | 10.47 |
| 120°+ | 481 | 1.88 | 3.01 | 10.64 | 10.47 |
stress of 0.27 N/mm² (tensile) at the shell edge ($\psi=0$) is significantly exceeded by the largest net (that is, total) meridional stress of 0.35 N/mm² (tensile) at about $\psi=2^\circ$ from the edge. However, these stress levels are relatively low, in relation to stress levels elsewhere, so that the stress-raising effect of the bending disturbance in the edge zone of the upper spherical closure will have little consequence on the design of the shell. Similarly, although the edge disturbance raises the membrane meridional stress from 0.27 N/mm² (tensile) to a net value of 0.33 N/mm² (tensile) at the upper edge ($\psi=0$) of the ogival shell, these levels of stress are soon exceeded by the rising membrane meridional tension in the ogival shell as $\phi$ increases beyond 60°.

At the lower edge of the ogival shell ($\phi=120^\circ$), the effect of the edge disturbance is to raise the membrane meridional stress from 2.47 N/mm² (compressive) at $\psi=0$ to a peak net value of 3.10 N/mm² (compressive) just away from the edge. Just across the lower junction at the edge of the lower spherical shell, the same membrane meridional compression of 2.47 N/mm² is raised to a peak net compression of 3.01 N/mm² by the bending disturbance. As noted for the upper junction, these levels of stress are soon exceeded by the rising membrane meridional compression in the lower spherical shell as $\phi$ increases beyond 120°.

Turning now to the hoop-stress variations, it is noted
that the membrane hoop stresses are generally much higher than the bending-disturbance contributions, for all edges. Thus, at the edge of the upper spherical shell, the effect of the edge disturbance is to raise the membrane hoop stress from a tensile value of 0.95 N/mm$^2$ to a tensile net value of only 1.09 N/mm$^2$, whereas at the upper edge of the ogival shell, the membrane hoop tension of 1.14 N/mm$^2$ is actually lowered to a tensile net peak of 1.09 N/mm$^2$. Similarly, at the lower edge of the ogival shell, the membrane hoop tension of 9.41 N/mm$^2$ is raised only marginally by the bending disturbance to a net value of 10.64 N/mm$^2$, while at the edge of the lower spherical shell, the membrane hoop tension of 11.05 N/mm$^2$ is actually lowered (albeit slightly) to a tensile net value of 10.64 N/mm$^2$.

5. Conclusions and recommendations

In conclusion, it may be stated that for the sizes and configurations of egg-shaped digestor shells that have been considered, junction edge effects are generally not as significant as membrane effects, and the design of the shell will be governed primarily by calculated membrane stresses. The discontinuity effects do alter the local meridional stresses significantly, but the net stresses that ensue are considerably lower than membrane stresses elsewhere, so that the stress-raising effect of the discontinuity effects have little bearing on the design of the shell. These conclusions may tentatively be extrapolated to larger egg-shaped digestor shells of the same basic configuration, but in order to properly investigate the effects of the scale of the structure on the stress levels in the shell, a parametric study should be undertaken. The closed-form analytical results for membrane and discontinuity stresses, as obtained in this study, would form a useful basis for such a study.

In order to control the noted steeply increasing membrane meridional compression (which may give rise to buckling problems in the thin shells of the type in question) and membrane hoop tension (which may give rise to cracking problems in the concrete) as one moves downwards from the lower junction of the digestor through the lower spherical closure towards the bottom, a number of measures are recommended. First, the thickness of the lower spherical closure should be increased rapidly with distance $s$ from the shell edge. This thickness enhancement may, for instance, be in accordance with a parabolic law $(t=t_o+k s^2)$, which would decrease the stresses almost as rapidly as the stress resultants $N_{\theta}^o$ and $N_{r}^o$ are rising. Above the lower junction $(\phi=\phi_o)$, and moving upwards, the shell thickness may continue to be decreased within the ogival shell in accordance with the law $r_J=a t_o$ proposed earlier, until $\phi=90^\circ$ is reached, beyond which the shell thickness may be kept constant in the upper half of the ogival shell and throughout the upper spherical closure. The reason for prescribing a constant minimum thickness in the upper half of the digestor shell is that stress levels in these regions are generally small to moderate.

Alongside the shell thickening recommended in the lower half of the digestor shell, tensile reinforcement must be provided, or prestressing adopted, and the levels of these progressively increased as one moves from the equatorial level of the digestor towards the bottom.

Based on the observations of the foregoing study, and as the final measure for controlling the steeply increasing membrane stresses in the lower spherical closure, it is recommended that the support ring of the digestor, or the edge of the shell-soil interactive surface in the case of the soil-supported digestor, be raised to 45$^\circ$ from the downward direction of the axis of revolution of the digestor shell (that is, $\phi=135^\circ$). This raising of the support level will cut off the very excessive membrane stresses that would otherwise occur towards the bottom of the tank.

Acknowledgements

The assistance of postgraduate students (R. Nonxuba, H. Jaufeerally, O. Ajayi, Q. Corner and L. Grunitz) with the preparation of the illustrations, and the generation of some of the numerical results, is gratefully acknowledged. Mr. V. Balden of the University of Cape Town also assisted with the FE modelling of the digestor shell.

References